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On state-dependent implication in quantum-mechanical distant correlations

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Abstract. Let ρ_{12} be an arbitrary given composite-system state (statistical operator). It is shown that same-subsystem events imply each other state-dependently according to ρ_{12} if and only if they act equally in the range of the corresponding subsystem state (reduced statistical operator). Opposite-subsystem events imply each other in the same way if and only if they are twin events, i.e. $E_1\rho_{12} = F_2\rho_{12}$. If ρ_{12} is a pure state, it is shown that the anti-unitary correlation operator also plays a decisive role in the latter implication.

1. Introduction

To begin with, we give a short summary of the concepts of the state-dependent implication in quantum logic that are required for this investigation.

Let $\mathcal{H}$ be the state space of a quantum system, and let $\mathcal{P}(\mathcal{H})$ be the set of all events (projectors) in $\mathcal{H}$. A (partial) order (i.e. a binary relation that is reflexive, transitive and antisymmetric (see Birkhoff 1940)) is defined in $\mathcal{P}(\mathcal{H})$ as follows

$$E \leqslant F \qquad \text{if } EF = E. \tag{1}$$

Equivalently, in terms of the ranges one has $R(E) \subseteq R(F)$.

Since the projectors can also be interpreted as propositions, one calls $\mathcal{P}(\mathcal{H})$ a quantum logic, having in mind that it is a partially ordered set with '$\leqslant$' given by (1), the so-called *absolute implication* contained in it.

Any other implication is a relative one. It is a pre-order (a binary relation that is reflexive and transitive, cf Birkhoff (1940)) in one of the Boolean subalgebras $\mathcal{B}$ of $\mathcal{P}(\mathcal{H})$, i.e. in the domain of definition of the relative implication. (For the necessity of restriction to $\mathcal{B}$ see Herbut (1994).) Any relative implication is defined by specifying an ideal Δ in $\mathcal{B}$, and by taking the corresponding factor algebra $\mathcal{B}/\Delta$ (cf Herbut 1994, 1995).

The equivalence relation (a binary relation that is reflexive, transitive and symmetric) '$\sim_\Delta$' redefines the classes in the sense that two events are equivalent if and only if they belong to the same class. In other words, denoting by $[E]$ the equivalence class (element of $\mathcal{B}/\Delta$) to which E belongs, $\forall E \in \mathcal{B}$, $[E] = \{F : F \sim_\Delta E\}$. One has

$$E \sim_\Delta F, \qquad E, F \in \mathcal{B} \qquad \text{if both } EF^\perp \in \Delta \qquad \text{and} \qquad E^\perp F \in \Delta \tag{2}$$

where $E^\perp \equiv 1 - E$ etc.

The *relative implication* '$\leqslant_\Delta$' can be entirely defined in terms of the equivalence relation '$\sim_\Delta$' given by (2) and the absolute implication '$\leqslant$' as follows:

$$E \leqslant_\Delta F,\ \ E, F \in \mathcal{B},\ \ \text{if } \exists G \in \mathcal{B} : G \sim_\Delta E,\ \ \text{and } \exists H \in \mathcal{B} : H \sim_\Delta F,\ \ \text{and } G \leqslant H. \tag{3a}$$

An important connection between '$\sim_\Delta$' and '$\leqslant_\Delta$' is given by

$$E \sim_\Delta F,\ \ E, F \in \mathcal{B},\ \ \textit{if and only if both}\ E \leqslant_\Delta F \text{ and } F \leqslant_\Delta E \tag{3b}$$

(see Herbut (1994)).

In this study we will be mostly interested in '$\sim_\Delta$', and we will refer to it as the relation of *mutual (relative) implication*.

The special case of *state-dependent implication* '$\leqslant_\rho$' is determined by any quantum state (statistical operator) ρ in $\mathcal{H}$. Let Q_0 be the null projector of ρ. Then

$$\Delta_\rho \equiv \{E : E \in \mathcal{B}, E \leqslant Q_0\} \tag{4a}$$

is an ideal in $\mathcal{B}$, and by definition

$$`\leqslant_\rho\text{'} = `\leqslant_{\Delta_\rho}\text{'} \tag{5}$$

(see Herbut (1994, 1995)). If the quantum state ρ is pure, i.e. $\rho = |\phi\rangle\langle\phi|$, then equivalently

$$\Delta_{|\phi\rangle} = \{E : E \in \mathcal{B}, E|\phi\rangle = 0\}. \tag{4b}$$

It is noteworthy that in general the quantum state is a mixture of pure states:

$$\rho = \sum_n w_n |\psi_n\rangle\langle\psi_n| \tag{6}$$

but the state-dependent implication '$\leqslant_\rho$' is determined only by its null projector

$$Q_0 = 1 - \sum_n |\psi_n\rangle\langle\psi_n| \tag{7}$$

(it is here assumed that (6) is a spectral form of ρ; otherwise, (6) can be more general).

Now we derive a result that we will make use of below.

Lemma 1. Let ρ be a state (statistical operator) and $\mathcal{B}$ a Boolean subalgebra of the quantum logic $\mathcal{P}(\mathcal{H})$ of a quantum system. Then two events (projectors) $E, F \in \mathcal{B}$ *imply each other state-dependently according to* ρ, i.e. $E \sim_\rho F$, *if and only if*

$$EQ = FQ \tag{8}$$

where Q is the *range projector* of ρ.

Proof. (a) We assume state-dependent equivalence of E and F according to ρ. In view of (2) and (4*a*), this takes the following explicit form:

$$E(1-F)(1-Q) = E(1-F)$$
$$(1-E)F(1-Q) = (1-E)F.$$

These relations simplify down to

$$EQ = EFQ \qquad FQ = EFQ$$

implying (8).

(b) We assume the validity of (8). Then $EQ = EFQ$, and, reading the above argument backwards, i.e. adding the required terms, we obtain

$$E(1-F)(1-Q) = E(1-F)$$

or symbolically, $EF^\perp \leqslant (1-Q)$. Symmetrically, (8) implies $FQ = EFQ$, and the addition of the required terms leads to $E^\perp F \leqslant (1-Q)$. In view of (2) and (4*a*), we have $E \sim_\rho F$. □

2. A summary of the canonical entities of distant correlations in mixed and pure composite-system states

Now, we outline some basic concepts of canonical distant-correlation theory, some of which were developed in previous work (Herbut and Vujičić 1976, Vujičić and Herbut 1984).

Any two-subsystem (e.g. two-particle) composite-system state $|\Psi\rangle_{12}$ ($\in (\mathcal{H}_1 \otimes \mathcal{H}_2)$) can be equivalently expressed as an antilinear Hilbert–Schmidt operator A_a mapping the state space (Hilbert space) $\mathcal{H}_1$ of the first subsystem into that of the other. Namely, these operators form one realization of the tensor product of the two subsystem spaces (see Gel'fand and Vilenkin (1964), Herbut and Vujičić (1976)).

To introduce the canonical entities, one writes A_a in terms of its polar factors (Herbut and Vujičić 1976):

$$|\Psi\rangle_{12} \Leftrightarrow A_a = U_a \rho_1^{1/2} \tag{9}$$

where

$$\rho_1 \equiv \mathrm{Tr}_2\, |\Psi\rangle_{12}\langle\Psi|_{12} \tag{10}$$

is the reduced statistical operator, physically the state, of the first subsystem. The operator U_a, the antilinear so-called correlation operator, maps the range $R(\rho_1)$ ($\subseteq \mathcal{H}_1$) of ρ_1 onto that of ρ_2 (the symmetrically defined reduced statistical operator or state of the second subsystem). Finally, 'Tr_2' denotes the partial trace in $\mathcal{H}_2$.

Opposite-subsystem events E_1 and F_2 in a given state $|\Phi\rangle_{12}$ satisfying

$$(E_1 \otimes 1)|\Phi\rangle_{12} = (1 \otimes F_2)|\Phi\rangle_{12} \tag{11}$$

are called twin events (cf Herbut and Vujičić (1976), Vujičić and Herbut (1984)) on account of the following two physical properties:

(i) The events E_1 and F_2 have the same probability of occurrence in $|\Phi\rangle_{12}$: $\langle\Phi|(E_1 \otimes 1)|\Phi\rangle_{12} = \langle\Phi|(1 \otimes F_2)|\Phi\rangle_{12}$.

(ii) The ideal occurrence (i.e. occurrence in ideal measurement) of E_1 or of F_2 in the state $|\Phi\rangle_{12}$ converts this state into one and the same state:

$$(E_1 \otimes 1)|\Phi\rangle_{12}/\langle\Phi|(E_1 \otimes 1)|\Phi\rangle_{12}^{1/2} = (1 \otimes F_2)|\Phi\rangle_{12}/\langle\Phi|(1 \otimes F_2)|\Phi\rangle_{12}^{1/2}.$$

Properties (11), (i) and (ii) generalize to a general (mixed or pure) state:

$$(E_1 \otimes 1)\rho_{12} = (1 \otimes F_2)\rho_{12} \tag{12}$$

(i′) Equal probability:

$$\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12} = \mathrm{Tr}_{12}(1 \otimes F_2)\rho_{12}$$

(ii′) Equal change-of-state:

$$(E_1 \otimes 1)\rho_{12}(E_1 \otimes 1)/\,\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12} = (1 \otimes F_2)\rho_{12}(1 \otimes F_2)/\,\mathrm{Tr}_{12}(1 \otimes F_2)\rho_{12}$$

(we have also utilized the adjoint of condition (12)).

Definition. We shall also call two opposite-subsystem events E_1 and F_2 twin events (projectors) in the case of a general (mixed or pure) state ρ_{12} if algebraic condition (12) (reducing to (11) in the case of a pure state) is satisfied.

3. Mutual state-dependent implication of same-subsystem events

Theorem 1. Let ρ_{12} be an arbitrary statistical operator in a composite-system state space $\mathcal{H}_1 \otimes \mathcal{H}_2$, let $\rho_i \equiv \mathrm{Tr}_j\, \rho_{12}$, $i, j = 1, 2$, $i \neq j$, be the reduced statistical operators implied by ρ_{12}, and let $\mathcal{B}_i$, be a given Boolean subalgebra of the quantum logic $\mathcal{P}(\mathcal{H}_i)$ of subsystem i, $i = 1, 2$. Then

$$\forall E_i, F_i \in \mathcal{B}_i : E_i \sim_{\rho_{12}} F_i \tag{13}$$

i.e. *two same-subsystem events state-dependently imply each other if and only if,* in terms of the range projectors Q_i of ρ_i,

$$E_i Q_i = F_i Q_i \tag{14}$$

are valid, $i = 1, 2$.

Proof. (a) Let (13) be valid. Then, both

$$E_i(1 - F_i) \leqslant (1 - Q_i) \qquad \text{and} \qquad (1 - E_i)F_i \leqslant (1 - Q_i) \tag{15a, b}$$

hold true (cf (2) and (4*a*)). On the one hand, the identity $E_i = E_i F_i + E_i(1 - F_i)$ gives $E_i Q_i = E_i F_i Q_i + E_i(1 - F_i)Q_i$. On the other hand, (15*a*) actually means that $E_i(1 - F_i) = E_i(1 - F_i)(1 - Q_i)$, which gives $0 = E_i(1 - F_i)Q_i$. Altogether, it follows that $E_i Q_i = E_i F_i Q_i$. The symmetric argument (in conjunction with the commutativity of E_i and F_i) leads to $F_i Q_i = E_i F_i Q_i$. Hence (14) is valid. (b) We assume the validity of (14). One has always $E_i(1 - F_i)(1 - Q_i) = E_i - E_i Q_i - E_i F_i + E_i F_i Q_i$. Relation (14) allows us to replace $F_i Q_i$ by $E_i Q_i$, implying

$$E_i(1 - F_i)(1 - Q_i) = E_i(1 - F_i).$$

This is the explicit form of (15*a*). The symmetrical argument establishes (15*b*). According to (2) and (4*a*), (13) is then valid. □

Corollary 1. In the notation of theorem 1, if one has $E_i \leqslant Q_i$, $F_i \leqslant Q_i$, then $E_i \sim_{\rho_{12}} F_i$ *if and only if* $E_i = F_i$, $i = 1, 2$.

Corollary 2. If $[E_i, \rho_i] = 0$, one has

$$\forall E_i \in \mathcal{B}_i \qquad E_i \sim_\rho Q_i E_{i_{12}} \qquad i = 1, 2.$$

(Note that here also $[E_i, Q_i] = 0$ since $(1 - Q_i)$ is the characteristic projector of ρ_i corresponding to the characteristic values zero.)

Corollary 3. In view of lemma 1 and theorem 1, we can say that first-subsystem events imply each other state-dependently according to ρ_{12} if and only if they do so according to ρ_1, i.e.

$$(\mathcal{B}_1 \otimes 1)/\Delta_{\rho_{12}} = (\mathcal{B}_1/\Delta_{\rho_1}) \otimes 1 \tag{16}$$

and symmetrically for the second subsystem.

Corollary 4. If $[\mathcal{B}_i, \rho_i] = 0$, then each equivalence class in $(\mathcal{B}_i/\Delta_{\rho_i})$ can be written as

$$\{E_i' + E_i'' : \text{fixed } E_i' \leqslant Q_i,\ \text{all } E_i'' \leqslant (1 - Q_i)\} \qquad i = 1, 2.$$

More light is thrown on the ramifications of theorem 1, and especially on corollaries 3 and 4, by the following result.

Lemma 2. One has $(E_1 \otimes 1) \leqslant (1 - Q_{12})$, where E_1 is an arbitrary projector in $\mathcal{H}_1$, and Q_{12} is the range projector of an arbitrary given statistical operator ρ_{12}, *if and only if* $E_1 \leqslant (1 - Q_1)$, where Q_1 is the range projector of $\rho_1 \equiv \mathrm{Tr}_2\, \rho_{12}$. The symmetrical result (obtained by interchanging the indices 1 and 2) is also valid.

Before we prove lemma 2, we establish an auxiliary result.

Lemma 3. If E and Q are two projectors and ρ is a statistical operator such that Q is its range projector, then $E \leqslant (1 - Q)$ is equivalent to

$$E\rho = 0. \tag{17}$$

(We had the special case $\rho \equiv |\phi\rangle\langle\phi|$ of this in (4*b*) above.)

Proof. (a) The assumed inequality actually claims that $E = E(1 - Q)$. Since always $E\rho = EQ\rho$, (17) follows.

(b) We assume the validity of (17). Let $\rho = \sum_i r_i |i\rangle\langle i|$ be a spectral form of ρ, $\forall i$: $r_i > 0$. Applying E to the characteristic relation $\rho|i\rangle = r_i|i\rangle$, and taking into account (17), one obtains: $\forall i : E|i\rangle = 0$. Hence, $E(1 - Q) = E(1 - \sum_i |i\rangle\langle i|) = E$, i.e. $E \leqslant (1 - Q)$ as claimed.

Proof of lemma 2. (a) Assuming $(E_1 \otimes 1) \leqslant (1 - Q_{12})$, we have, according to lemma 3, $(E_1 \otimes 1)\rho_{12} = 0$. Then $E_1\rho_1 = E_1\,\mathrm{Tr}_2\,\rho_{12} = \mathrm{Tr}_2(E_1 \otimes 1)\rho_{12} = 0$. (We made use of a partial-trace identity, cf the first relation in (19) below.) This implies, on account of lemma 3, $E_1 \leqslant (1 - Q_1)$.

(b) Assuming $E_1 \leqslant (1-Q_1)$, we have $E_1\rho_1 = 0$ (cf theorem 3). Then also $E_1\rho_1E_1 = 0$. Hence,

$$\mathrm{Tr}_2(E_1 \otimes 1)\rho_{12}(E_1 \otimes 1) = 0$$

(here we also utilized the last partial-trace identity in (19) below). This implies $\mathrm{Tr}_{12}(E_1 \otimes 1)\rho_{12}(E_1 \otimes 1) = 0$. A positive operator has zero trace only if it is zero. Hence, $(E_1 \otimes 1)\rho_{12}(E_1 \otimes 1) = 0$. A lemma of Lüders (1951) claims that if $C^\dagger BC = 0$, where B is positive and C linear, then $BC = 0$. In our case, $(E_1 \otimes 1)\rho_{12} = 0$ or $(E_1 \otimes 1) \leqslant (1 - Q_{12})$ (cf lemma 3). □

4. Mutual state-dependent implication of opposite-subsystem events

Theorem 2. Assuming the notation of theorem 1, two opposite-subsystem events (projectors) E_1 and F_2 imply each other state-dependently according to a given composite-system state (statistical operator) ρ_{12} *if and only if* (12) is valid, i.e. if and only if E_1 and F_2 are *twin events* in the state ρ_{12}.

Proof. (a) We assume $(E_1 \otimes 1) \sim_{\rho_{12}} (1 \otimes F_2)$. In view of (2), (4*a*) and lemma 3, this amounts to

$$((1 - E_1) \otimes 1)(1 \otimes F_2)\rho_{12} = 0 = (E_1 \otimes 1)(1 \otimes (1 - F_2))\rho_{12}. \tag{18}$$

Hence, one obtains

$$(1 \otimes F_2)\rho_{12} = (E_1 \otimes F_2)\rho_{12} = (E_1 \otimes 1)\rho_{12}.$$

(b) Relation (12) implies

$$((1 - E_1) \otimes 1)\rho_{12} = (1 \otimes (1 - F_2))\rho_{12}$$

and then also (18). □

An important necessary condition for twin events (in the general case) is their compatibility with the corresponding states. More precisely, one has the following result.

Proposition. If E_1 and F_2 are twin events (projectors) with respect to a given composite-system state (statistical operator) ρ_{12}, i.e. if (12) is valid, then

$$[E_1, \rho_1] = 0 \qquad [F_2, \rho_2] = 0$$

i.e. the events are compatible with the corresponding states $\rho_i \equiv \mathrm{Tr}_j\, \rho_{12}$ $(i, j = 1, 2, i \neq j)$ of the subsystems.

Proof. We make use of the following easily checked partial-trace identities:

$$A_1 \,\mathrm{Tr}_2\, B_{12} = \mathrm{Tr}_2(A_1 \otimes 1)B_{12} \qquad \mathrm{Tr}_2(1 \otimes C_2)B_{12} = \mathrm{Tr}_2\, B_{12}(1 \otimes C_2) = (\mathrm{Tr}_2\, B_{12})C_2. \quad (19)$$

Thus, utilizing (12) and its adjoint, one derives

$$\begin{aligned} E_1 \,\mathrm{Tr}_2\, \rho_{12} &= \mathrm{Tr}_2(E_1 \otimes 1)\rho_{12} = \mathrm{Tr}_2(1 \otimes F_2)\rho_{12} = \mathrm{Tr}_2\, \rho_{12}(1 \otimes F_2) \\ &= \mathrm{Tr}_2\, \rho_{12}(E_1 \otimes 1) = (\mathrm{Tr}_2\, \rho_{12})E_1. \end{aligned}$$

The symmetrical argument completes the proof. □

Theorem 3. Let $|\Phi\rangle_{12}$ be an arbitrary state vector of a composite two-subsystem system. Let $\rho_i \equiv \mathrm{Tr}_j\, |\Phi\rangle_{12}\langle\Phi|_{12}$, $i, j = 1, 2$, $i \neq j$, be the reduced statistical operators with Q_i, $i = 1, 2$ as the corresponding range projectors. Let, further, U_a be the antilinear correlation operator implied by $|\Phi\rangle_{12}$ (cf (9)). Let, finally, $\mathcal{B}_1$ be a given Boolean subalgebra of first-subsystem events, and $\mathcal{B}_2$ one of second-subsystem events such that

$$\mathcal{B}_2 \supseteq \{F_1' + F_2'' : F_2' = U_a E_1 U_a^{-1} Q_2, E_1 \in \mathcal{B}_1, [E_1, Q_1] = 0; F_2'' \leqslant (1 - Q_2)\}$$

and $\mathcal{B}_{12}$ as the Boolean subalgebra of $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ spanned by $\mathcal{B}_1 \times \mathcal{B}_2$, i.e. as the minimal structure of this kind containing the latter set. Then, one has

$$(E_1 \otimes 1) \sim_{|\Phi\rangle_{12}} (1 \otimes F_2) \qquad E_1 \in \mathcal{B}_1, F_2 \in \mathcal{B}_2 \qquad (20)$$

if and only if both

$$[E_1, \rho_1] = 0 \qquad (21)$$

and

$$F_2' \equiv F_2 Q_2 = U_a E_1 U_a^{-1} Q_2 \qquad (22)$$

are valid.

Proof. (a) We assume that (20) is valid. Then, according to theorem 2, the events E_1 and F_2 are twins regarding $|\Phi\rangle_{12}$. Further, according to the proposition, E_1 and ρ_1 commute. Hence, we can take a characteristic orthonormal basis $\{|\phi_m\rangle_1 : \forall m\}$ of ρ_1 and simultaneously of E_1 spanning the range $R(Q_1)$. Let the corresponding characteristic values be $\{r_m : \forall m\}$ and $\{e_m : \forall m\}$ respectively. Then, $\{(U_a|\phi_m\rangle_1)_2 : \forall m\}$ is an orthonormal basis spanning the range $R(Q_2)$ (cf Herbut and Vujičić (1976)). Besides, $|\Phi\rangle_{12}$ can be written in the so-called Schmidt canonical form in terms of the introduced entities (Herbut and Vujičić 1976, relation (32)):

$$|\Phi\rangle_{12} = \sum_m r_m^{1/2} |\phi_m\rangle_1 \otimes (U_a|\phi_m\rangle_1)_2. \qquad (23)$$

We replace this in (11), and we take the partial scalar product with $\langle\phi_m|_1$ (with a fixed but arbitrary m value). One obtains:

$$\forall m : e_m (U_a|\phi_m\rangle_1)_2 = F_2 (U_a|\phi_m\rangle_1)_2$$

($r_m^{1/2}$, which is necessary positive, cancels on both sides). On the other hand, one obviously has

$$U_a E_1 U_a^{-1}(U_a|\phi_m\rangle_1)_2 = e_m(U_a|\phi_m\rangle_1)_2.$$

Hence, $F_2 = U_a E_1 U_a^{-1}$ in $R(Q_2)$, i.e. (22) is valid.

(b) Assuming the validity of (21) and of (22), and utilizing (23) with $\{|\phi_m\rangle_1 : \forall m\}$ as a common characteristic basis of ρ_1 and of E_1, one obtains

$$(1 \otimes F_2)|\Phi\rangle_{12} = \sum_m r_m^{1/2} e_m |\phi_m\rangle_1 \otimes (U_a|\phi_m\rangle_1)_2.$$

On the other hand, on account of (23), $(E_1 \otimes 1)|\Phi\rangle_{12}$ gives the same. Thus E_1 and F_2 are twin events, and, according to theorem 2, they imply each other state-dependently according to $|\Phi\rangle_{12}$.

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